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SPIN AND STATISTICS OF QUANTUM KINKS

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Apart from some brief and inconclusive remarks concerning the problem of spin and statistics of quantum kinks in space-time dimension $D > 2$, we give a detailed discussion of the $D = 2$ situation. Our main result is that two-dimensional quantum kinks are statistical "schizons"; they exist in the same Hilbert space either as bosons or as fermions. In those cases where one can introduce local kink-sector generating operators as in the sine-Gordon model, the Bose and Fermi fields are strictly local fields, which are relatively non-local with respect to each other.

1. Introduction

The traditional framework of quantum field theory is based on local Lagrangians containing fields which carry internal quantum numbers. Local conservation laws in this framework arise *via* Noether's theorem. Field commutation relation ((anti)-commutation, para field-commutation, para field-commutation) are converted *via* LSZ asymptotics into the corresponding particle statistics. Particle statistics therefore in this traditional setting is rigorously derivable to the extent of the validity of LSZ asymptotics, i.e. for short ranged interactions.

Recent theoretical ideas have transcended this traditional framework in two opposing directions. Whereas the phenomenon of charge screening [1] (or its more dramatic version, the idea of particle confinement) leads to vanishing charges of the would-be Noether symmetry currents and therefore to the absence of the apparent symmetry on the level of physical states, the study of quantum kinks [2] yields charge sectors [3] ** which have their origin in the topological aspects of the Lagran-

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** A unified study of confinement and kinks for a class of two-dimensional field theories can be found in ref. [4].

gian systems. The fundamental problem of these "hidden" topological symmetries is the construction of interpolating fields (i.e. fields which generate the various topological charge sectors) for the new particles * carrying topological charge. Here, in *principle*, two different situations may arise: it may be possible to find a local interpolating field (i.e. (anti)-commuting with itself for space-like separations), or there may be only quasilocal interpolating fields.

An example of the first situation is the sine-Gordon [5] model, whereas the kink of the A_2^4 Higgs model seems to belong to the second situation (sect. 3). Note that the extended nature of classical kink solutions does not shed any direct light on this quantum field theoretical problem; its direct consequence is a non-local relative commutation structure between the basic Lagrangian field and the kink field. From the experience with QED, for which a local electron operator in a physical (positive definite, i.e. Coulomb-gauge) Hilbert space does not exist, one would expect that there are no local interpolating operators for 't Hooft-Polyakov [6,7] monopoles or dyons. A still more difficult question is the spin and statistics problem of particles belonging to the non-trivial charge sectors [8–10]. Here the only clear-cut result we have to offer is in $D = 2$ for kink models with a mass gap as in the sine-Gordon and the A_2^4 model. In sects. 2 and 3 we will show that such particles are "schizons" from the point of view of statistics and spin. The main difficulty in the $D > 2$ monopole models (sect. 4) is the complicated connection between interpolating fields and particles, i.e. the breakdown of the LSZ asymptotics, due to the appearance of "infraparticles"

2. General remarks on spin and statistics in two dimensions

It is a well-known fact that there is no intrinsic physical meaning of spin in two-dimensional relativistic theories. What one normally calls "spin" in this case is the Lorentz spin, i.e. the transformation properties of the wave function or field operators under Lorentz transformations. As long as there exist one-particle states one can (trivially) carry through Wigner's [11] famous analysis for two-dimensional space time to conclude that one-particle states can always be chosen to transform as scalars:

$$U(\Lambda)|p^0, p^1\rangle = |\cosh\chi p^0 + \sinh\chi p^1, \cosh\chi p^1 + \sinh\chi p^0\rangle = |\Lambda p\rangle\,. \tag{1}$$

Clearly one is free to introduce another equivalent description

$$|p^0, p^1\rangle_s = \left(\frac{p^0 + p^1}{m}\right)^s |p^0, p^1\rangle\,, \tag{2}$$

* We follow the traditional terminology of using the word soliton only for systems with an infinite number of non-trivial conservation laws and using the work kink for the topological charge sectors and their affiliated particles.

leading to a "spin" s transformation law

$$U(\Lambda)\,|p\rangle_s = e^{s\chi}\,|\Lambda p\rangle\,. \tag{3}$$

This freedom of assignment of two-dimensional Lorentz spin corresponds in $D = 4$ to the well-known fact that with one Wigner representation there are many relativistic wave equations transforming differently under the Lorentz group.

In the zero mass case, as for the massless Thirring fields, the Lorentz spin enters the representation theory (i.e. the Casimir operator) of the conformal group [12,13]. For a massive theory, however, the "spin" assignment is entirely a matter of convention. This is well known. What is perhaps at first sight more surprising is that, as explained below, in two dimensions the statistics is also a matter of convention. As long as a field theory describes ordinary particles one means by statistics the asymptotic (in the LSZ sense) free-particle statistics. Suppose for definiteness that those particles were bosons:

$$|p\rangle = a^{+}(p)\,|0\rangle\,,$$

$$[a(p), a^{+}(p')] = p^{0}\delta(p - p')\,,$$

$$[a(p), a(p')] = 0 = [a^{+}(p), a^{+}(p')]\,. \tag{4}$$

Consider now

$$b^{+}(p) = a^{+}(p)\exp\{\; i\pi \int_{p}^{\infty} n(p')\,\mathrm{d}p'\}\,, \tag{5a}$$

$$n(p) = \frac{1}{p^{0}}\,a^{+}(p)\,a(p)\,. \tag{5b}$$

The b's satisfy canonical anticommutation relations, i.e. they are fermion operators. This simply means that there is a one-to-one mapping between antisymmetric and symmetric p-space wave functions

$$f_{\mathrm{A}}(p, p') = \epsilon(p - p')\,f_{\mathrm{S}}(p, p')\,, \tag{6}$$

which allows one to interpret any bosonic state in terms of fermions and *vice versa*. Although in *higher* dimensions *similar* mappings can be introduced, they do not share with eqs. (5) the property of being Lorentz invariant. Note that formula (5a), which makes sense only for operators in an infinite volume, has its lattice theory counterparts in the Pauli-Jordan transformation which transforms "paulions" (i.e. objects which only commute at different lattice points) into lattice fermions [14].

One could be formally tempted to introduce in two dimensions a *generalized* statistics (not parastatistics!) by inserting into the exponent of the right-hand side of eq. (5a) an arbitrary real number $0 \leqslant s < 2$. However, if one demands in accordance with general principles that the Fourier transform of the momentum space wave function describes probabilities of (approximate) position measurements, the simultaneous requirement

$$|\tilde{f}(x, x')|^2 = |\tilde{f}(x', x)|^2, \qquad |f(p, p')|^2 = |f(p', p)|^2 \tag{7}$$

restricts our choice of s to be 0 or 1. Our assertion that the assignment of Bose or Fermi statistics to the particle states of a given two-dimensional quantum field theory is entirely conventional seems to contradict the well-known fact (valid even in a two-dimensional world) that a periodic table of elements requires fermions.

The apparent paradox is resolved by realizing that to find the energy levels of an atom with a given local potential one needs an at least approximate notion of localization. Allowing for the highly non-local interactions induced by the mapping (6) one can have a Bose system exhibiting the same energy levels as a Fermi system with local interactions.

In the next section we will investigate this problem of statistics (and "spin") for two-dimensional kinks. In the case of the sine-Gordon kink, where the existence of a local sector creating Heisenberg fields is known [5], we will show that they can be chosen (in the same Hilbert space) either as local $s = \frac{1}{2}$ anticommuting or as local $s = 0$ commuting fields. A typical case of a kink for which a non-local (quasi-local) kink-generating field has been found is the two-dimensional A^4 theory. Here the choice of a "quasi-local Fermi field" instead of a quasi-local Bose field will be shown to be possible. They are interpolating fields for the same one-particle states whose corresponding multiparticle states will obey either Fermi or Bose statistics.

3. Two examples of two-dimensional kinks

What is the consequence of the general two-dimensional spin and statistics situation of the last section for quantum kinks *? Here one should distinguish two classes of theories, namely those for which one only has quasi-local operators leading from the vacuum to the one-kink sectors or models for which strictly local sector-generating operators exist. As a possible example for the first class we discuss the A^4 theory in two dimensions **. Here a formal expression for such an operator would be

$$\hat{\psi}_f = \exp\{i\pi \int f(x) : (\pi^2(x, t) + A^2(x, t)) : \mathrm{d}x\}, \tag{8}$$

* See the first footnote in the introduction.

** Our treatment of the A^4 quantum kink deviates from that of Fröhlich [15]. We do not understand this construction based on a doubling of field components and a subsequent restriction of the bigger algebra.

with

$$\pi(x, t) = \frac{\partial}{\partial t} A(x, t) ,$$

and $f(x)$ any function which is 0 on the left-hand side of R^1 space and 1 on the right-hand side. Such a transformation describes a space-dependent phase-space rotation, which is the identity on the left-hand side and the transformation

$$\begin{pmatrix} A \\ \pi \end{pmatrix} \to \begin{pmatrix} -A \\ -\pi \end{pmatrix} \tag{9}$$

on the right-hand side. However, the density in eq. (8) has an incurable ultraviolet problem, which forces one to work with the quasi-local fields:

$$q(x) = (m^2 + \Delta)^{1/4} A(x), \qquad p(x) = (m^2 + \Delta)^{-1/4} \pi(x) , \tag{10}$$

i.e.

$$\psi_f = \exp\{i\pi \int f(x) : (p^2(x) + q^2(x)) : \mathrm{d}x\} . \tag{11}$$

This operator has asymptotic commutation relations

$$\lim_{a\to\infty} [\psi_f, U(a)\, \psi_f U^+(a)] = 0 . \tag{12}$$

The application of the Haag-Ruelle [18] theory leads to kink states with Bose statistics. On the other hand, an operator of the form

$$B_{f,g} = \psi_f A_g, \quad A_g = \int A(x) g(x)\, \mathrm{d}x , \tag{13}$$

with $g \in D$, a smooth localized test function yields an asymptotic anticommutation relation

$$\lim_{a\to\infty} \{B_{f,g}, U(a)\, B_{f,g} U^+(a)\} = 0 . \tag{14}$$

The argument for eq. (14) is the following: the region of effective variation of the test function f is shifted by $U(a)$ to the far right-hand side. Hence the $U(a)\, \psi_f U^+(a)$ (approximately) commutes with the A_g from the first factor. The $U(a) A_g U^+(a)$ from the second factor, however, anticommutes with the ψ_f from the first term.

Quasi-classical arguments suggest [2] that the lowest energy state in the non-vacuum sector is a one-particle state at rest. Since the field A has a mass gap (i.e. no infrared problems) this quasi-classical argument is credible in the A^4 quantum field theory. There is no reason, on the basis of conservation laws, to suspect that the commuting field ψ_f has a different behaviour, with respect to this new state $|p\rangle$, from the $B_{f,g}$, i.e. we expect

$$\langle 0|\psi_f|p\rangle \neq 0\,, \tag{15}$$

$$\langle 0|B_{f,g}|p\rangle \neq 0\,. \tag{16}$$

As we argued in sect. 2, the simultaneous existence of an interpolating quasi-local Bose field and a quasi local Fermi field for the same particle is completely consistent in two dimensions.

The more interesting case is the sine-Gordon case, where the existence of a covariant local interpolating field for the sine-Gordon kinks is known [5]. It can be chosen to be the anticommuting massive Thirring spinor. We now show that a local commuting charged Bose field (with Lorentz spin $s = 0$) can be explicitly constructed by controllable space-time limiting procedures. The starting point is the exponential of the axial current potential [16]:

$$B_\lambda(x) = \mathcal{N}[\mathrm{e}^{2i\sqrt{\pi}\lambda\varphi(x)}]\,, \qquad j_{\mu 5} = \frac{1}{\sqrt{\pi}}\,\partial_\mu\varphi\,. \tag{17}$$

Consider now the short-distance Wilson expansion of ψ with B_λ. In order to keep the computations explicit and simple, let us restrict our consideration to the massive free Dirac model. In this case [16]

$$B_\lambda(x) = {}^*_*\,\mathrm{e}^{L_\lambda(x)}\,{}^*_*\,, \tag{18}$$

where the double stars refer to free fermion ordering:

$$L_\lambda(x) = -\frac{\sin\pi\lambda}{2\pi}\left[\int\frac{\mathrm{e}^{-2\lambda\hat\theta}}{\cosh\hat\theta}\left(\mathrm{e}^{i(p+q)x}a_p^+b_q^+ + \mathrm{e}^{-i(p+q)x}a_qb_p\right)\mathrm{d}\theta_p\,\mathrm{d}\theta_q\right.$$

$$\left.+\int\frac{\mathrm{e}^{-2\lambda\hat\theta}}{\sinh\hat\theta}\left(\mathrm{e}^{i\pi\lambda}\mathrm{e}^{i(p-q)x}a_p^+a_q + \mathrm{e}^{-i\pi\lambda}\mathrm{e}^{-i(p-q)x}b_q^+b_p\right)\mathrm{d}\theta_p\,\mathrm{d}\theta_q\right], \tag{19}$$

where $\hat\theta = \frac{1}{2}(\theta_p - \theta_q)$. Consider first the short distance expansion of $\psi_1(x)$ with $B_\lambda(0)$:

$$\psi_1(x)B_\lambda(0) = -\frac{1}{\sqrt{2\pi}}\,\frac{\sin\pi\lambda}{2\pi}\,\frac{m}{2}\,{}^*_*\int a(p')\,\mathrm{e}^{-ip'x}\,\mathrm{e}^{-\theta_{p'}/2}\,\mathrm{d}\theta_{p'}$$

$$\times\int\left(a_p^+b_q^+\,\frac{\mathrm{e}^{-2\lambda\hat\theta}}{\cosh\hat\theta} - a_p^+a_q\,\frac{\mathrm{e}^{-2\lambda\hat\theta}}{\sinh\hat\theta}\,\mathrm{e}^{i\pi\lambda}\right)\mathrm{d}\theta_p\,\mathrm{d}\theta_q\,B_\lambda(0)\,{}^*_*$$

$$+\ \text{non-singular terms}\,. \tag{20}$$

The two-fermion contraction terms can be decomposed into a leading and a non-leading contribution. The integrand is rapidity space for a given $|\lambda|$ has the strongest

θ_p increase for positive λ:

$$\frac{e^{-(\lambda+1/2)\theta_p}}{\cosh\hat{\theta}} = 2e^{-\lambda\theta_p}e^{-\theta_p/2} + R(\theta_p)\,, \tag{21}$$

with integrable $R(\theta_p)$ for $-\lambda < 1$.

The contribution from the integrable R can be combined with the regular terms. Using the representation

$$\left(\frac{t+x+i\epsilon}{t-x+i\epsilon}\right)^{\nu/2} K_\nu(m\sqrt{x^2-t^2}) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-i(t\cosh\theta - x\sinh\theta)m - \nu\theta}\, d\theta\,, \tag{22}$$

we obtain for short distances ($\lambda > 0$)

$$\psi_1(x)B_\lambda(0) = \text{const}\left(\frac{t+x+i\epsilon}{t-x+i\epsilon}\right)^{\lambda/2} K_\lambda(m\sqrt{x^2-t^2})$$

$$\times \,{}^*_*\!\int (b^+(q)\,e^{\theta_q(\lambda-1/2)} + e^{i\pi\lambda}a(q)\,e^{\theta_q(\lambda-1/2)})\,d\theta_q\,B_\lambda{}^*_* + \text{reg. terms}\,. \tag{23}$$

The spin and dimension of the finite operator multiplying the singular factor are

$$s = \tfrac{1}{2} - \lambda\,, \qquad d = \tfrac{1}{2} + \lambda^2 - |\lambda|\,.$$

In an anlogous manner we obtain for the second component a singular term for $\lambda < 0$

$$\psi_2(x)B_\lambda(0) = \text{const}\left(\frac{t+x+i\epsilon}{t-x+i\epsilon}\right)^{\lambda/2} K_{|\lambda|}(m\sqrt{x^2-t^2})$$

$$\times \,{}^*_*\!\int (b^+(q)\,e^{\theta_q(\lambda+1/2)} + e^{i\pi\lambda}a(q)\,e^{\theta_q(\lambda+1/2)})\,d\theta_q B_\lambda{}^*_* + \text{reg. terms}\,. \tag{24}$$

Clearly by choosing $\lambda = \frac{1}{2}$ for the first component and $\lambda = -\frac{1}{2}$ for the second component we obtain two (kink-) charge-carrying scalar operators

$$O_1(0) = {}^*_*\!\int (b^+(q)\,e^{-i\pi/4} + a(q)\,e^{i\pi/4})\,d\theta_q B_{1/2}{}^*_*\,, \tag{25a}$$

$$O_2(0) = {}^*_*\!\int (b^+(q)\,e^{i\pi/4} + a(q)\,e^{-i\pi/4})\,d\theta_q B_{-1/2}{}^*_*\,. \tag{25b}$$

Clearly these operators have spin zero and they commute for space-like distances, since they are obtained by a limiting procedure from $B_{\pm 1/2}(x)$ which anticommute with the ψ's for space-like distances [16].

4. Problems in higher dimensions

In a space-time of dimension larger than two, kinks in renormalizable quantum field theories have to be discussed in the context of gauge theories [3]. Although one expects the usual connection between spin and statistics, there are a number of points to be clarified.

Firstly, the usual proofs of spin statistics (cf. Wightman-Streater [19]), as well as the structural investigation of the statistics of sectors [20], rely on the assumption of locality (or quasi-locality) of the interpolating fields. In a gauge theory, as long as one works in a physical Hilbert space (no ghosts) locality can only be taken for granted for observable gauge-invariant operators.

The intrinsically non-local nature of charge-carrying fields is clearly illustrated by the fact that a q-number gauge transformation of the second kind, which ought to leave the physical content of the theory invariant, can completely change the commutation relations of the fields.

In fact, starting with an anticommuting charged field

$$\{\psi(x)\,\psi(y)\}_{ET} = \{\psi(x)\,\psi^+(y)\}_{ET} = 0\,, \qquad x \neq y\,, \tag{26}$$

and introducing a transformation analogous to the one employed in sect. 3, viewed as a gauge transformation

$$\widetilde{\psi}(x) = \exp\{i\pi \int j^0(x')\,\theta(x^1 - x'^1)\,\mathrm{d}^3x'\}\cdot\psi(x)\,, \tag{27}$$

one finds

$$[\widetilde{\psi}(x), \widetilde{\psi}(y)]_{ET} = [\widetilde{\psi}(x), \widetilde{\psi}^+(y)]_{ET} = 0, \quad x \neq y\,. \tag{28}$$

One would believe that this ambiguity of field statistics is not in this case (contrary to what happens in two dimensions) reflecting itself in a corresponding ambiguity of particle statistics. Here one is immediately led to another problem: the fact that for gauge theories in four dimensions the usual mass-shell description of particles fails and both an infrared clouding as well as Coulomb distortion must be taken into account [7].

A clear understanding of these points seems to us essential in order to associate a certain statistics to dyons of the kind recently obtained by Hasenfratz and 't Hooft [8], Jackiw and Rebbi [9], and Goldhaber [10].

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